OLS regression interpretation

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1 Level-level regression

Consider the following model:

$$y = \beta_1 + \beta_2 x + u \tag{1}$$

with $y = (y_1, ..., y_N)$ the dependant variable, or regressor, $x = (x_1, ..., x_N)$ the independent variable, or regressand, and $u = (u_1, ..., u_N)$ the error terms. β_1 and β_2 are the parameters to be estimated. The fitted values of y are expressed as follows:

$$\hat{y} = \hat{\beta}_1 + \hat{\beta}_2 x$$

To interpret $\hat{\beta}_2$, we find the derivative of $\hat{y}$ with respect to x.

$$\begin{aligned} d\hat{y} &= \frac{\partial \hat{y}}{\partial x} dx \\ d\hat{y} &= \hat{\beta}_2 dx \end{aligned}$$

For small changes, $\Delta x \approx dx$ with $\Delta x = x' - x$ the change in x. Then, we can write:

$$\Delta \hat{y} = \hat{\beta}_2 \Delta x$$

If x changes by 1 unit, i.e. $\Delta x = 1$, y is expected to change by $\hat{\beta}_2$ units, since $\Delta \hat{y} = \hat{\beta}_2 \times 1 = \hat{\beta}_2$.

2 Level-log regression

Consider the following model:

$$y = \beta_1 + \beta_2 \log(x) + u \tag{2}$$

The change of $\hat{y}$ due to a change in x can be computed as follows:

$$\begin{aligned} d\hat{y} &= \frac{\partial \hat{y}}{\partial x} dx \\ d\hat{y} &= \hat{\beta}_2 \frac{1}{x} dx \\ 100 \times d\hat{y} &= \hat{\beta}_2 \frac{dx}{x} \times 100 \\ 100 \times d\hat{y} &= \hat{\beta}_2 \% \Delta x \end{aligned}$$

Note that for small changes, $\%\Delta x \approx \frac{dx}{x} \times 100$, which is the growth rate of x expressed in percentage. Then, we can write:

$$\Delta \hat{y} = \frac{\hat{\beta}_2}{100} \% \Delta x$$

If x changes by 1%, i.e. $\%\Delta x = 1$, y is expected to change by $\hat{\beta}_2/100$ units, since $\Delta \hat{y} = \frac{\hat{\beta}_2}{100} \times 1 = \frac{\hat{\beta}_2}{100}$.

3 Log-level regression

Consider the following model:

$$\log(y) = \beta_1 + \beta_2 x + u \tag{3}$$

In this case, a slightly different method can be applied. Consider the fitted values $\log(\hat{y}) = \hat{\beta}_1 + \hat{\beta}_2 x$ and $\log(\hat{y}') = \hat{\beta}_1 + \hat{\beta}_2 x'$. Then, we can write:

$$\begin{aligned}
\log(\hat{y}') - \log(\hat{y}) &= \hat{\beta}_1 + \hat{\beta}_2 x' - (\hat{\beta}_1 + \hat{\beta}_2 x) \\
\Delta \log(\hat{y}) &= \hat{\beta}_1 + \hat{\beta}_2 x' - \hat{\beta}_1 - \hat{\beta}_2 x \\
\Delta \log(\hat{y}) &= \hat{\beta}_2 (x' - x) \\
\Delta \log(\hat{y}) &= \hat{\beta}_2 \Delta x
\end{aligned}$$

For small changes, $\Delta \log(\hat{y}) \approx \frac{\Delta \hat{y}}{y}$.[1] Then, we have:

$$\begin{aligned}
\frac{\Delta \hat{y}}{y} &= \hat{\beta}_2 \Delta x \\
100 \times \frac{\Delta \hat{y}}{y} &= \hat{\beta}_2 \Delta x \times 100 \\
\% \Delta \hat{y} &= \hat{\beta}_2 \Delta x \times 100
\end{aligned}$$

If x changes by 1 unit, i.e. $\Delta x = 1$, y is expected to change by $\hat{\beta}_2 \times 100$ %, since $\%\Delta \hat{y} = \hat{\beta}_2 \times 1 \times 100 = \hat{\beta}_2 \times 100$.

Another way to find the result more precisely is by using the exponential transformation as follows:

$$\begin{aligned}
\log(\hat{y}') - \log(\hat{y}) &= \hat{\beta}_1 + \hat{\beta}_2 x' - (\hat{\beta}_1 + \hat{\beta}_2 x) \\
\log(\frac{\hat{y}'}{\hat{y}}) &= \hat{\beta}_2 (x' - x) \\
\frac{\hat{y}'}{\hat{y}} &= exp(\hat{\beta}_2 \Delta x) \\
\frac{\hat{y}'}{\hat{y}} - 1 &= exp(\hat{\beta}_2 \Delta x) - 1 \\
\frac{\hat{y}' - \hat{y}}{\hat{y}} &= exp(\hat{\beta}_2 \Delta x) - 1 \\
100 \times \frac{\Delta \hat{y}}{\hat{y}} &= [exp(\hat{\beta}_2 \Delta x) - 1] \times 100
\end{aligned}$$

[1] For a small a, we can write $\log(a+1) \approx a$. Then, $\log(\frac{\hat{y}'}{\hat{y}}) = \log(\frac{\hat{y}'}{\hat{y}} - 1 + 1) = \log(\frac{\hat{y}' - \hat{y}}{\hat{y}} + 1) \approx \frac{\hat{y}' - \hat{y}}{\hat{y}} = \frac{\Delta \hat{y}}{\hat{y}}$. Note that $\log(\frac{\hat{y}'}{\hat{y}}) = \log(\hat{y}') - \log(\hat{y})$.

$$\%\Delta\hat{y} = [exp(\hat{\beta}_2\Delta x) - 1] \times 100$$

If x changes by 1 unit, i.e. $\Delta x = 1$, y is expected to change by $[exp(\hat{\beta}_2) - 1] \times 100$ %.

4 Log-log regression

Consider the following model:

$$\log(y) = \beta_1 + \beta_2 \log(x) + u \tag{4}$$

Similarly to the previous section, we can compute:

$$\begin{aligned}
\log(\hat{y}') - \log(\hat{y}) &= \hat{\beta}_1 + \hat{\beta}_2 \log(x') - (\hat{\beta}_1 + \hat{\beta}_2 \log(x)) \\
\Delta \log(\hat{y}) &= \hat{\beta}_2(\log(x') - \log(x)) \\
\Delta \log(\hat{y}) &= \hat{\beta}_2 \Delta \log(x) \\
\frac{\Delta\hat{y}}{y} &= \hat{\beta}_2 \frac{\Delta\hat{x}}{x} \\
100 \times \frac{\Delta\hat{y}}{y} &= \hat{\beta}_2 \frac{\Delta\hat{x}}{x} \times 100 \\
\%\Delta\hat{y} &= \hat{\beta}_2 \%\Delta\hat{x}
\end{aligned}$$

If x changes by 1%, i.e. $\%\Delta x = 1$, y is expected to change by $\hat{\beta}_2$%.